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D. B. Karp & E. G. Prilepkina

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Applications of the Stieltjes and Laplace transform representations of the hypergeometric functions

D. B. Karp \(^a,b\) and E. G. Prilepkina \(^a,b\)

\(^a\)Far Eastern Federal University, Vladivostok, Russia; \(^b\)Institute of Applied Mathematics, FEBRAS, Vladivostok, Russia

**ABSTRACT**

In our previous work, we found sufficient conditions to be imposed on the parameters of the generalized hypergeometric function in order that it be completely monotonic or of Stieltjes class. In this paper we collect a number of consequences of these properties. In particular, we find new integral representations of the generalized hypergeometric functions, evaluate a number of integrals of their products, compute the jump and the average value of the generalized hypergeometric function over the branch cut \([1, \infty)\), and establish new inequalities for this function in the half-plane \(\Re z < 1\). Furthermore, we discuss integral representations of absolutely monotonic functions and present a curious formula for a finite sum of products of gamma ratios as an integral of Meijer’s \(G\) function.

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1. Introduction and preliminaries

Throughout the history of the hypergeometric functions, integral representations played an important role in their study. The celebrated Euler’s integral for the Gauss functions \( _2F_1 \) \([1, \text{Theorem 2.2.1}]\) was probably the first among them. More recently, Kiryakova \([2, \text{Chapter 4}]\) observed that the generalized hypergeometric functions possess similar representations with densities expressed via Meijer’s \(G\) function. These representations have been extended and massively applied in our papers \([3–6]\). For the generalized hypergeometric functions of the Gauss and Kummer type, the most important integral representations are those given by the (generalized) Stieltjes and Laplace transforms, respectively. If the density in the corresponding representation is nonnegative, then the resulting hypergeometric function is completely monotonic or belongs to the (generalized) Stieltjes class.

The purpose of this note is to share some observations about completely monotonic and Stieltjes functions and illustrate them by the hypergeometric examples. As a by-product of these observations we evaluate a number of integrals involving the generalized hypergeometric functions which are neither contained in the most comprehensive collection \([7]\) nor are solvable by \textit{Mathematica}. We further find the jump and the average value of the generalized hypergeometric function over the branch cut and present some inequalities.
for this function resulting from the Stieltjes representation. Some facts presented in this note complement and/or illustrate the results of the recent work by Koumandos and Pedersen [8].

We now turn to the details. Let us fix some notation and terminology first. The standard symbols \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) will be used to denote the natural, real and complex numbers, respectively; \( \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{R}_+ = [0, \infty) \). A nonnegative function \( f \) defined on \((0, \infty)\) is called completely monotonic if it has derivatives of all orders and \((-1)^n f^{(n)}(x) \geq 0\) for \( n \geq 0 \) and \( x > 0 \). A function \( \phi : [0, \infty) \to \mathbb{R} \) is called absolutely monotonic if it is infinitely differentiable on \([0, \infty)\) and \( \phi^{(k)}(x) \geq 0 \) for all \( k \geq 0 \) and all \( x \geq 0 \). According to Widder [9, Theorem 3a], an absolutely monotonic function \( \phi \) on \([0, \infty)\) has an extension to an entire function with the power series expansion \( \phi(z) = \sum_{n \geq 0} \phi_n z^n \), where \( \phi_n \geq 0 \) for all \( n \geq 0 \). As usual \( \Gamma(z) \) stands for Euler’s gamma function and \( (a)_n = \Gamma(a + n)/\Gamma(a) \) denotes the rising factorial (or Pochhammer’s symbol). Throughout the paper we will use the shorthand notation for the products and sums. For \( a \in \mathbb{C} \) define

\[
\Gamma(a) = \Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_p), \quad (a)_n = (a_1)_n(a_2)_n \cdots (a_p)_n, \\
a + \mu = (a_1 + \mu, a_2 + \mu, \ldots, a_p + \mu), \quad a > 0 \iff a_k > 0 \quad \text{for} \quad k = 1, \ldots, p;
\]

(\(a)_1\) will be further simplified to \((a) = \prod_{k=1}^{p} a_k\). We will write \( a_{[k]} \) for the vector \(a\) with \(k\)th element removed, i.e. \( a_{[k]} = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_p)\). Further, we will follow the standard definition of the generalized hypergeometric function \( {}_pF_q \) as the sum of the series

\[
{}_pF_q \left( \begin{array}{c} a \\ b \end{array} \bigg| z \right) = {}_pF_q(a; b; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_p)_n}{(b_1)_n(b_2)_n \cdots (b_q)_n n!} z^n, \quad (1.1)
\]

if \( p \leq q, z \in \mathbb{C} \). If \( p = q + 1 \) the above series has unit radius of convergence and \( {}_pF_q(z) \) is defined as analytic continuation of its sum to \( \mathbb{C} \setminus \{1, \infty\} \). Here \( a = (a_1, \ldots, a_p) \) and \( b = (b_1, \ldots, b_q) \) are (generally complex) parameter vectors, such that \(-b_j \notin \mathbb{N}_0, j = 1, \ldots, q\).

According to Karp [3, Corollary 1]

\[
{}_pF_p \left( \begin{array}{c} a \\ b \end{array} \bigg| -z \right) = \frac{\Gamma(b)}{\Gamma(a)} \int_{0}^{1} e^{-zt} G_{p, p}^{0,0} \left( t \bigg| \begin{array}{c} b \\ a \end{array} \right) \frac{dt}{t} \quad (1.2)
\]

and

\[
{}_{p+1}F_p \left( \begin{array}{c} a \\ b \end{array} \bigg| -z \right) = \frac{\Gamma(b)}{\Gamma(a)} \int_{0}^{\infty} e^{-zt} G_{p, p+1}^{0,1,0} \left( t \bigg| \begin{array}{c} b \\ a \end{array} \right) \frac{dt}{t} \quad (1.3)
\]

under convergence conditions of the integrals on the right-hand side, as elucidated in [3, Theorem 1]. Here \( G_{p,q}^{m,n} \) denotes Meijer’s \( G \) function whose definition and properties can be found in [3, §2, 5, Section 2, 7, Section 8.2, 10, Section 16.17, 11, Section 12.3]. Formulas (1.2) and (1.3) combined with Bernstein’s theorem imply that the hypergeometric functions on the left-hand sides are completely monotonic if the \( G \) functions in the integrands are nonnegative. According to [3, Theorem 2] the function \( G_{p, p}^{0,0} \) from (1.2) is
nonnegative if the Müntz polynomial

\[ v_{a,b}(t) = \sum_{k=1}^{p} (t^{a_k} - t^{b_k}) \geq 0 \quad \text{for } t \in [0,1]. \]  

Note that the vector \( a \) in (1.3) contains \( p+1 \) elements, while \( v_{a,b}(t) \) is defined for two real vectors \( a, b \) of equal size. Therefore, \( v_{a,b}(t) \) in (1.4) should be replaced by \( v_{a[p+1],b}(t) \) for \( G_{p,p+1}^{p+1,0} \) to be nonnegative, while the remaining parameter \( a_{p+1} \) may take arbitrary positive values. See [4, Proof of Theorem 8] for details. Inequality (1.4) is implied by the stronger condition \( b \prec W a \) known as the weak supermajorization [12, section 2] and given by Marshall et al. [13, Definition A.2]

\[
0 < a_1 \leq a_2 \leq \cdots \leq a_p, \quad 0 < b_1 \leq b_2 \leq \cdots \leq b_p, \\
\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for } k = 1, 2, \ldots, p. 
\]  

Further sufficient conditions for the validity of (1.4) in terms of \( a, b \) can be found in our recent paper [12, Section 2].

This paper is organized as follows. Section 2 is concerned with the Laplace transform representations and their corollaries, such as integral representations of absolutely monotonic functions and new evaluations of convolution integrals containing hypergeometric functions. Sections 3 is devoted to the consequences of the (generalized) Stieltjes transform representation for the Gauss type generalized hypergeometric functions. These are: new integrals for \( \binom{p}{p+1} _{1} F_{1}(z) \) in the left half-plane and the sector \( \arg(-z) \in (-\pi/3, \pi/3) \), a formula for the jump of this functions over the branch cut \((\infty, -1]\) and for its mean on the banks of the cut, inequalities for this function in the half-plane \( \Re(z) < 1 \) implied by its univalence. Finally, in the ultimate Section 4 we present a curious formula for the finite sum of the differences of ratios of the gamma products as an integral of \( G \) function over the interval \((0, 1)\).

2. Laplace transform representations and their consequences

According to Karp [3, Theorem 4], the function

\[ x \mapsto x^{-a_{p+1}} F_p \left( \binom{a}{b} \bigg| -1/x \right) \]

is completely monotonic if \( a > 0 \) and \( v_{a[p+1],b}(t) \geq 0 \) for \( t \in [0,1] \). Using our recent result from [4], we can prove a similar statement for \( F_p \). Recall that \( a[k_1,k_2,\ldots,k_r] \) denotes the vector \( a \) with the elements \( a_{k_1}, a_{k_2}, \ldots, a_{k_r} \) removed.

**Theorem 2.1:** Suppose \( p \geq 2, \ a, b \in \mathbb{R}^p \) are ascending positive vectors and set \( a_*= (a_{[1,p]}, 3/2) \) and \( b_* = b_{[1]} \). Assume that \( 0 < a_1 \leq 1, b_1 \geq a_1 + 1 \) and \( v_{a_*,b_*}(t) \geq 0 \) on \([0,1] \).
Then the function
\[ x \mapsto x^{-a_p}F_p(a; b; -1/x) \]
is completely monotonic.

**Proof:** Termwise integration confirms that

\[ x^{-a_p}F_p(a; b; -1/x) = \frac{1}{\Gamma(a_p)} \int_0^\infty e^{-xt}t^{a_p-1}p-1F_p(a[p]; b; -t) \, dt. \]

Under conditions of the theorem the function \( p-1F_p(a[p]; b; -t) \) is positive on \([0, \infty)\) by Karp and López [4, Theorem 7] and the claim follows by Bernstein's theorem. ■

It is immediate from the definition of a completely monotonic function that a linear combination of such functions with nonnegative coefficients is completely monotonic. As pointwise limit of a sequence of completely monotonic functions is also completely monotonic [9, p.151], we conclude that \( \phi(1/x) \) is absolutely monotonic (see also [14, Theorem 3]). We summarize these facts in the next simple observation.

**Theorem 2.2:** Suppose \( \phi : [0, \infty) \mapsto \mathbb{R} \) is absolutely monotonic. Then \( \phi(1/x) \) is completely monotonic and its representing measure is given by \( \nu(dt) = [\phi_0\delta_0 + v(t)] \, dt \), where \( v(t) = \sum_{k \geq 0} \phi_{k+1}t^k/k! \). If, furthermore, \( \{k!\phi_k\}_{k \geq 0} \) forms a Stieltjes moment sequence, then \( \phi(-x) \) is also completely monotonic and its representing measure \( \mu(dt) \) is related to \( v(t) \) by

\[ v(t) = \int_{[0, \infty)} \left( \frac{\mu}{t} \right)^{1/2} I_1(2\sqrt{ut}) \mu(du), \]

where \( I_1 \) is the modified Bessel function of the first kind.

**Proof:** Indeed, according to Widder [9, Theorem 2b], the function \( x \mapsto \phi(1/x) \) is completely monotonic. Furthermore,

\[
\int_{[0, \infty)} e^{-xt}[\phi_0\delta_0 + v(t)] \, dt = \phi_0 + \sum_{k \geq 0} \frac{\phi_{k+1}}{k!} \int_{[0, \infty)} e^{-xt}t^k \, dt
\]

\[ = \phi_0 + \sum_{k \geq 0} \frac{\phi_{k+1}}{k!} \frac{\Gamma(k+1)}{x^{k+1}} = \sum_{k \geq 1} \frac{\phi_k}{x^k} + \phi_0 = \phi(1/x), \]

which proves the first claim. On the other hand, by the Stieltjes moment property \( k!\phi_k = \int_{[0, \infty)} t^k \mu(dt) \) for some nonnegative measure \( \mu \), so that

\[ \phi(-x) = \sum_{k=0}^{\infty} \phi_k(-x)^k = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \int_{[0, \infty)} t^k \mu(dt) = \int_{[0, \infty)} e^{-xt} \mu(dt). \]
Then,
\[ v(t) = \sum_{k \geq 0} \frac{\phi_{k+1} t^k}{k!} = \sum_{k \geq 0} \frac{t^k}{k!(k + 1)!} \int_{[0, \infty)} u^{k+1} \mu(du) \]
\[ = \int_{[0, \infty)} u \sum_{k \geq 0} \frac{(ut)^k}{(2)k!} \mu(du) = \int_{[0, \infty)} u_0 F_1 (-; 2; ut) \mu(du) \]
\[ = \int_{[0, \infty)} \left( \frac{u}{t} \right)^{1/2} I_1 (2 \sqrt{ut}) \mu(du), \]
where we have used the series expansion for the modified Bessel function [10, Formula (10.25.2)].

**Remark 2.1:** Theorem 2.2 shows that the absolutely monotonic functions on \([0, \infty)\) are precisely those that are representable in the form
\[ \phi(y) = C + \int_0^\infty e^{-t/y} v(t) \, dt, \]
where \(C \geq 0\) and \(v(t)\) is an entire function of minimal exponential type (i.e. of order \(< 1\) or of order \(= 1\) and minimal type) with nonnegative Taylor coefficients \(v_k\). The minimal exponential type simply means that \(\lim \sup_{k \to \infty} [v_k k!]^{1/k} = 0\) [15, section 1.3]. Other integral representations can be obtained using [9, Theorem 2b] which asserts that \(\phi(f(x))\) is completely monotonic once \(\phi\) is absolutely monotonic and \(f(x)\) is completely monotonic. The class of the representing measures will depend on \(f(x)\). Taking, for instance, \(f(x) = e^{-x}\) yields \(\phi(y) = \int_{[0, \infty)} y^t \mu(dt)\). This is, of course, just another way of writing the Taylor series expansion, so that \(\mu(dt)\) must be supported on nonnegative integers with mass concentrated at \(n\) decreasing fast enough. Note that the problem of finding an integral representation for absolutely monotonic functions has been considered in [16].

**Remark 2.2:** The density \(v(t)\) of the representing measure for \(\phi(1/x)\) in Theorem 2.2 is also absolutely monotonic. Laplace transforms of absolutely monotonic functions have been recently studied by Koumandos and Pedersen [8, Theorem 1.1]. The proof of their theorem indicates that the authors essentially consider the functions of the form \((1/x)\phi(1/x)\). This shows that Theorem 2.2 complements [8, Theorem 1.1].

According to Theorem 2.2 the function \(x \mapsto p F_q(a; b; 1/x)\) is completely monotonic for \(p \leq q\) and any positive parameter vectors \(a, b\). We write the representing measure explicitly in the next corollary.

**Corollary 2.1:** Suppose \(p \leq q + 1\), \(a\) and \(b\) are any complex vectors, \(b\) does not contain non-positive integers. Then the following identity holds:
\[ f(x) = p F_q(a; b; 1/x) = \int_{[0, \infty)} e^{-xt} \left[ \frac{\binom{a + 1}{b + 1, 2}}{\binom{a + 1}{b + 1, 2}} F_{p+1} \right] dt, \quad (2.1) \]
where \(\delta_0\) is the Dirac measure with mass 1 concentrated at zero. Here \(x > 0\) if \(p \leq q\) and \(x > 1\) if \(p = q + 1\).
**Proof:** Straightforward termwise integration yields the claim. ■

Curiously enough, formula (2.1) is different from the related integral evaluations [1, Exercise 11, p.115, 7, (2.22.3.1)].

Further, by application of [8, Theorem 1.1] to the function \( f \) defined in (2.1) the derivatives \((-1)^k(x^k f(x))^{(k)}\) are completely monotonic for each \( k \in \mathbb{N}_0 \). Moreover, if \( R_n(y) \) is \( n \)-th Taylor remainder for \( pF_q(a; b; y) \) then \( R_n(1/x) \) is completely monotonic of order \( n \) (\( g : (0, \infty) \mapsto \mathbb{R} \) is completely monotonic of order \( \alpha > 0 \) if \( x \mapsto x^\alpha g(x) \) is completely monotonic). These properties are relatively straightforward in the case of the above \( f \), but they may serve as a good illustration for [8, Theorem 1.1]. A slightly different hypergeometric illustration is given in [8, Remark 4.3].

**Remark 2.3:** From what we have said both \( pF_q(a; b; -x) \) and \( pF_q(a; b; 1/x) \) are completely monotonic if we assume that \( v_{a,b}(t) = \sum_{j=1}^q (t^{a_j} - t^{b_j}) \geq 0 \) for \( t \in [0, 1] \). If, in addition, conditions of Theorem 2.1 are satisfied, then all three functions

\[
x \mapsto pF_q(a; b; -x), \quad x \mapsto pF_q(a; b; 1/x), \quad x \mapsto x^{-a} pF_q(a; b; -1/x)
\]

are completely monotonic.

Next, we want to combine (2.1) with the product formulas for hypergeometric functions (conveniently collected, for instance, in [17]) and employ Laplace convolution theorem to get new integral evaluations for hypergeometric functions. First we handle the general case. Suppose we have an identity of the form \( \phi_1(y) \phi_2(y) = \phi_0(y) \), where all three functions are absolutely monotonic and the representing measures in Theorem 2.2 are \( v_1(t) \, dt + \delta_0 \), \( v_2(t) \, dt + \delta_0 \) and \( v_0(t) \, dt + \delta_0 \), respectively (we assume without loss of generality that the constant terms in their Taylor expansions are all equal to 1). Note that \( v_i(t), \; i \in \{0, 1, 2\} \) vanish for \( t < 0 \). Then, by Widder [9, Chapter VI, section 10, formula (1)], the convolution of the measures of this special form is given by

\[
\int_{[0,t]} (v_0(u) + \delta_0) \, du = \int_{[0,t]} \left( \int_0^{t-u} \phi_1(x) \, dx + \mathbf{1}_{[u \leq t]} \right) (v_2(u) + \delta_0) \, du \\
= \int_{[0,t]} \phi_2(u) \left( \int_0^{t-u} \phi_1(x) \, dx \right) \, du + \int_{[0,t]} \phi_2(u) \, du \\
+ \int_{[0,t]} \left( \int_0^{t-u} \phi_1(x) \, dx + \mathbf{1}_{[u \leq t]} \right) \delta_0 \, du \\
= \int_0^t \phi_2(u) \left( \int_0^{t-u} \phi_1(x) \, dx \right) \, du \\
+ \int_0^t \phi_2(u) \, du + \int_0^t \phi_1(x) \, dx + \mathbf{1}_{[0 \leq t]}.
\]

Taking derivatives on both sides and using \( \partial_t \mathbf{1}_{[0 \leq t]} = \delta_0 \) and

\[
\partial_t \int_0^t \phi_2(u) \left( \int_0^{t-u} \phi_1(x) \, dx \right) \, du = \int_0^t \phi_2(t-u) \phi_1(u) \, du,
\]
we finally obtain
\[ v_0(t) = \int_0^t v_2(t-u)v_1(u) \, du + v_1(t) + v_2(t). \] (2.2)

Combined with the Laplace convolution theorem [9, Theorem 16a] these observations can be summarized as follows.

**Lemma 2.1:** Suppose \( x \rightarrow \phi_i(x) \), \( \phi_i(0) = 1 \), are absolutely monotonic for \( i \in \{0, 1, 2\} \) and are related by \( \phi_0(x) = \phi_1(x)\phi_2(x) \). If \( \phi_i(1/x) = \int_{[0,\infty]} e^{-xt}(v_i(t) + \delta_0) \, dt \) for \( i \in \{0, 1, 2\} \) in accordance with Theorem 2.2, then the representing functions \( v_i, i \in \{0, 1, 2\} \), are related by (2.2).

The following examples illustrate the application of this lemma to derive several new integral evaluations for products of hypergeometric functions. We believe them to be new. They are also inaccessible for Mathematica. These examples may be viewed as a complement to [18, Section 4.24.5], containing a number of similar but still different integrals. A general technique for evaluating the integrals of special functions has been developed in [19].

**Example 2.1:** Write the identity \((1-x)^{-a}(1-x)^{-b} = (1-x)^{-a+b}\) in the form
\[ _1F_0 \begin{pmatrix} \frac{a}{x} \\ _1F_0 \begin{pmatrix} \frac{b}{x} \\ _1F_0 \begin{pmatrix} \frac{a+b}{x} \\ \end{pmatrix} \end{pmatrix} \end{pmatrix} = _1F_0 \begin{pmatrix} \frac{a+b}{1/x} \end{pmatrix}. \]

Applying (2.1) to each term above we get explicit expressions for the representing measures \( \nu_i, i = 0,1,2 \). Substituting them into (2.2) yields
\[
ab \int_0^t _1F_1 \begin{pmatrix} \frac{a+1}{2} \\ \frac{b+1}{2} \end{pmatrix} \, du
= (a+b)_1F_1 \begin{pmatrix} \frac{a+b}{2} \\ t \end{pmatrix} - a_1F_1 \begin{pmatrix} \frac{a+1}{2} \\ t \end{pmatrix} - b_1F_1 \begin{pmatrix} \frac{b+1}{2} \\ t \end{pmatrix}.
\]

**Example 2.2:** Write Euler transformation [1, Theorem 2.2.5] in the form
\[ _1F_0 \begin{pmatrix} \frac{a+b-c}{1/x} \end{pmatrix} _2F_1 \begin{pmatrix} \frac{c-a, c-b}{c} \\ _1F_0 \begin{pmatrix} \frac{a, b}{x} \end{pmatrix} \end{pmatrix} = _2F_1 \begin{pmatrix} \frac{a, b}{1/x} \end{pmatrix}. \]

Applying (2.1) to each term above we get explicit expressions for the representing measures \( \nu_i, i = 0,1,2 \). Substituting them into (2.2) yields
\[
(a+b-c) \int_0^t _1F_1 \begin{pmatrix} \frac{a+b-c+1}{2} \\ \frac{c-a+1, c-b+1}{c+1, 2} \end{pmatrix} \, du
= \frac{ab}{(c-a)(c-b)} \_2F_2 \begin{pmatrix} \frac{a+1, b+1}{c+1, 2} \\ \frac{c(a+b-c)}{(c-a)(c-b)}_1F_1 \begin{pmatrix} \frac{a+b-c+1}{2} \\ t \end{pmatrix} - 2_2F_2 \begin{pmatrix} \frac{c-a+1, c-b+1}{c+1, 2} \\ t \end{pmatrix}.
\]
Example 2.3: Take the celebrated Clausen’s identity [17, (11)] in the form
\[
\left[ {\binom{a-1,b-1}{a+b-3/2}1/x} \right]^2 = 3F_2 \left( \begin{array}{c} 2a-2,2b-2,a+b-2 \\ a+b-3/2,2a+2b-4 \end{array} \mid 1/x \right).
\]
Writing the representing measure for each function in this formula by (2.1) and applying (2.2) we obtain after elementary manipulations
\[
\frac{(a-1)(b-1)}{2a+2b-3} \int_0^t 2F_2 \left( \begin{array}{c} a,b \\ a+b-1/2,2 \\ t-u \end{array} \mid u \right) 2F_2 \left( \begin{array}{c} a,b \\ a+b-1/2,2 \\ t \end{array} \mid \right) du
= 3F_3 \left( \begin{array}{c} 2a-1,2b-1,a+b-1 \\ a+b-1/2,2a+2b-3,2 \\ t \end{array} \mid \right) - 2F_2 \left( \begin{array}{c} a,b \\ a+b-1/2,2 \\ t \end{array} \mid \right).
\]

Example 2.4: Next, we take the product formula for the Bessel functions in the form [17, (18)]
\[
0F_1 \left( \begin{array}{c} - \\ a \end{array} \mid 1/x \right) 0F_1 \left( \begin{array}{c} - \\ b \end{array} \mid 1/x \right) = 2F_3 \left( \begin{array}{c} (a+b)/2,(a+b-1)/2 \\ a,b,a+b-1 \\ 4/x \end{array} \mid \right).
\]
As the argument on the right-hand side is 4/x, we need a slightly modified form of (2.1) given by
\[
\binom{p}{q} (a;b;\alpha/x) = \int_{[0,\infty]} e^{-xt} \left[ \alpha (a) b \binom{p}{q+1} (a+1,2 \mid \alpha t) + \delta_0 \right] dt
\]
and obtained by the obvious change of variable. This formula combined with (2.2) yields
\[
\frac{1}{ab} \int_0^t 0F_2 \left( \begin{array}{c} - \\ a+1,2 \\ t-u \end{array} \mid \begin{array}{c} - \\ b+1,2 \\ u \end{array} \right) du
= \frac{(a+b)}{ab} 2F_4 \left( \begin{array}{c} (a+b)/2+1,(a+b-1)/2+1 \\ a+1,b+1,a+b,2 \\ 4t \end{array} \mid \right) - \frac{1}{a} 0F_2 \left( \begin{array}{c} - \\ a+1,2 \\ t \end{array} \mid \right)
- \frac{1}{b} 0F_2 \left( \begin{array}{c} - \\ b+1,2 \\ t \end{array} \mid \right).
\]

Example 2.5: Using the same algorithm, Orr’s identities [17, (14)–(16)] lead to the next three integral evaluations
\[
ab \int_0^t 2F_2 \left( \begin{array}{c} a+1,b+1 \\ a+b+1/2,2 \\ t-u \end{array} \mid \begin{array}{c} a+1,b+1 \\ a+b+3/2,2 \\ u \end{array} \right) du
= 2(a+b) 3F_3 \left( \begin{array}{c} 2a+1,2b+1,a+b+1 \\ a+b+3/2,2a+2b,2 \\ t \end{array} \mid \right)
- (a+b+1/2) 2F_2 \left( \begin{array}{c} a+1,b+1 \\ a+b+1/2,2 \\ t \end{array} \mid \right)
- (a+b-1/2) 2F_2 \left( \begin{array}{c} a+1,b+1 \\ a+b+3/2,2 \\ t \end{array} \mid \right).
\]
3. Stieltjes transform representation and its consequences

Recall that a function \( f : (0, \infty) \to \mathbb{R} \) is called a Stieltjes function, if it is of the form

\[
f(x) = c + \int_{(0,\infty)} \frac{\mu(dt)}{x + t},
\]

where \( c \) is a nonnegative constant and \( \mu \) is a positive measure on \([0, \infty)\) such that the above integral converges for any \( x > 0 \), see [9, Chapter VIII, 20, p.294]. In [5, Theorem 8] we found sufficient conditions on parameters in order that the function \( p_{+1}F_p(a; b; -x) \) be of this class. This allows the use of the Stieltjes inversion formula [9, Theorem 7a] to calculate the jump of the generalized hypergeometric function \( p_{+1}F_p(z) \) over the branch cut \([1, \infty)\). We can then employ an analytic continuation formula for the generalized hypergeometric function to get rid of the restrictions on parameters. Notwithstanding the simplicity of the calculation that follows, we are unaware of any references for formula (3.1).

**Theorem 3.1:** Suppose \( x > 1 \) and \( a, b \) are real vectors. Then the following identities hold true

\[
p_{+1}F_p \left( \begin{array}{c} a \\ b \end{array} \mid x + i0 \right) - p_{+1}F_p \left( \begin{array}{c} a \\ b \end{array} \mid x - i0 \right) = 2\pi i \frac{\Gamma(b)}{\Gamma(a)} G^{p+1,0}_{p+1,p+1} \left( \begin{array}{c} 1 \\ a \end{array} \mid 1, b \right), \tag{3.1}
\]

\[
\frac{p_{+1}F_p(a; b; x + i0) + p_{+1}F_p(a; b; x - i0)}{2} = -\frac{\pi}{\sqrt{x}} \frac{\Gamma(b)}{\Gamma(a)} G^{p+1,1}_{p+2,p+2} \left( \begin{array}{c} 1 \\ x \end{array} \mid 1/2, 1, b - 1/2 \right). \tag{3.2}
\]
Proof: According to Karp and Prilepkina [5, Theorem 8], the next formula holds

$$p+1F_p\left(\frac{a}{b} \left| -z \right. \right) = \int_1^\infty \frac{\rho(x) \, dx}{x + z},$$

for $0 < a_1 \leq 1, \ b \prec_W a_{[1]}$, where

$$\rho(x) = \frac{\Gamma(b)}{\Gamma(a)} \, G^{p+1,0}_{p+1,p+1} \left( \frac{1}{x} \left| \begin{array}{c} 1, b \end{array} \right. \right).$$

According to the Stieltjes inversion formula [9, Theorem 7a] or [21, (12)]

$$\rho(x) = \frac{1}{2\pi i} \left\{ p+1F_p \left( \frac{a}{b} \left| x + i0 \right. \right) - p+1F_p \left( \frac{a}{b} \left| x - i0 \right. \right) \right\}$$

and the claim follows under the above restrictions on parameters. To get rid of these restrictions we can apply [10, Formula (16.8.6)] reading

$$p+1F_p \left( \frac{a}{b} \left| x \right. \right) = \sum_{j=1}^{p+1} \frac{\Gamma(a_{[j]} - a_j) \Gamma(b)}{\Gamma(a_{[j]}) \Gamma(b - a_j)} (-x)^{-a_j} p+1F_p \left( \frac{a_j, 1 - b + a_j}{1 - a_{[j]} + a_j} \left| \frac{1}{x} \right. \right). \quad (3.3)$$

Since $(-x - i0)^{-a_j} - (-x + i0)^{-a_j} = 2ix^{-a_j} \sin(\pi a_j)$ for $x > 1$, we get

$$p+1F_p \left( \frac{a}{b} \left| x + i0 \right. \right) - p+1F_p \left( \frac{a}{b} \left| x - i0 \right. \right)$$

$$= 2i \sum_{j=1}^{p+1} \frac{\Gamma(a_{[j]} - a_j) \Gamma(b)}{\Gamma(a_{[j]}) \Gamma(b - a_j)} x^{-a_j} \sin(\pi a_j) p+1F_p \left( \frac{a_j, 1 - b + a_j}{1 - a_{[j]} + a_j} \left| \frac{1}{x} \right. \right).$$

The identity

$$2i \sum_{j=1}^{p+1} \frac{\Gamma(a_{[j]} - a_j) \Gamma(b)}{\Gamma(a_{[j]}) \Gamma(b - a_j)} x^{-a_j} \sin(\pi a_j) p+1F_p \left( \frac{a_j, 1 - b + a_j}{1 - a_{[j]} + a_j} \left| \frac{1}{x} \right. \right)$$

$$= 2\pi i \frac{\Gamma(b)}{\Gamma(a)} \, G^{p+1,0}_{p+1,p+1} \left( \frac{1}{x} \left| \begin{array}{c} 1, b \end{array} \right. \right)$$

following from [22, (2.4)] by Euler’s reflection formula completes the proof of formula (3.1). To demonstrate (3.2) we apply

$$(-x - i0)^{-a_j} + (-x + i0)^{-a_j} = 2x^{-a_j} \cos \pi a_j = -\frac{2\pi x^{-a_j}}{\Gamma(a_j - 1/2) \Gamma(3/2 - a_j)}$$

together with formula (3.3) to get (assuming $x > 1$)

$$p+1F_p(a; b; x + i0) + p+1F_p(a; b; x - i0)$$

$$= \frac{-\pi}{\sqrt{x}} \sum_{j=1}^{p+1} \frac{\Gamma(a_{[j]} - a_j) \Gamma(b)(1/x)^{a_j-1/2}}{\Gamma(a_{[j]}) \Gamma(b - a_j) \Gamma(a_j - 1/2) \Gamma(3/2 - a_j)}$$
\[
\times p + 2 \left. F_{p+1} \left( \begin{array}{c} a, b \\ x + i0 \\
\end{array} \right| 1 - b + a_j, a_j, a_j - 1/2 \right| 1/x \right).
\]

To verify that this expression is equal to the right-hand side of (3.2) it remains to apply \([10, 16.17.2]\) or \([7, 8.2.2.3]\).

As a corollary we recover \([10, (15.2.3)]\):

**Corollary 3.1:** Suppose \(x > 1\). Then the following identity holds

\[
2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| x + i0 \right) - 2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| x - i0 \right) = \frac{2\pi i \Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(1 + c - a - b)} (x - 1)^{c-a-b} 2 F_1 \left( \begin{array}{c} c - a, c - b \\ c - a - b + 1 \end{array} \right| 1 - x \right).
\]

**Proof:** Indeed, writing (3.1) for \(p = 1\) gives

\[
2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| x + i0 \right) - 2 F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| x - i0 \right) = 2\pi i \frac{\Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(2,2)} \left( \frac{1}{x} \right) \left( \frac{1,c}{a,b} \right).
\]

Further expressing \(G^{2,0}_{2,2}(1/x)\) via \(2 F_1\) according to \([22, p.4]\) and using Pfaff’s transformation \([10, (15.8.1)]\) we arrive at (3.4).

**Remark 3.1:** To verify the consistency of formulas (3.1) and (3.2) with the boundary values of the generalized hypergeometric function given in \([23]\) introduce the notation \((x > 1)\):

\[
F_+ = \left. p + 1 F_p \left( \begin{array}{c} a, b \\ c \end{array} \right| x + i0 \right), \quad F_- = \left. p + 1 F_p \left( \begin{array}{c} a, b \\ c \end{array} \right| x - i0 \right)
\]

for the values on the top and bottom banks of the branch cut, respectively. It follows from Theorem 3.1 that

\[
F_+ = - \frac{\pi \Gamma(b)}{\sqrt{\pi} \Gamma(a)} \left. G^{p+1,1}_{p+2,p+2} \left( \begin{array}{c} 1/2, 1, b - 1/2 \\ 1/x, a - 1/2, 1 \end{array} \right) \right| a, b - 1/2, 1 \right) + \pi i \frac{\Gamma(b)}{\Gamma(a)} \left. G^{p+1,0}_{p+1,p+1} \left( \begin{array}{c} 1 \\ 1/x \end{array} \right) \right| a, b - 1/2, 1 \right)
\]

and

\[
F_- = - \frac{\pi \Gamma(b)}{\sqrt{\pi} \Gamma(a)} \left. G^{p+1,1}_{p+2,p+2} \left( \begin{array}{c} 1/2, 1, b - 1/2 \\ 1/x, a - 1/2, 1 \end{array} \right) \right| a, b - 1/2, 1 \right) - \pi i \frac{\Gamma(b)}{\Gamma(a)} \left. G^{p+1,0}_{p+1,p+1} \left( \begin{array}{c} 1 \\ 1/x \end{array} \right) \right| a, b - 1/2, 1 \right).
\]

For real values of parameters these formulas give real and imaginary parts of the generalized hypergeometric function on the banks of the branch cut. By definition of Meijer’s \(G\) function we have

\[
- \frac{\pi \Gamma(b) \pi}{\sqrt{\pi} \Gamma(a)} \left. G^{p+1,1}_{p+2,p+2} \left( \begin{array}{c} 1/2, 1, b - 1/2 \\ 1/x, a - 1/2, 1 \end{array} \right) \right| a, b - 1/2, 1 \right)
\]

\[
= - \frac{\pi \Gamma(b) \pi}{2\pi i \sqrt{\pi} \Gamma(a)} \left. \int_{C'} \frac{\Gamma(-t + 1/2)\Gamma(t + a - 1/2)}{\Gamma(1 + t)\Gamma(-t)\Gamma(t + b - 1/2)} \left( \frac{1}{x} \right)^{-t} dt \right| a, b - 1/2, 1 \right)
\]

\[
= \frac{\pi \Gamma(b) \pi}{2\pi i \sqrt{\pi} \Gamma(a)} \left. \int_{C'} \frac{\sin \pi t \Gamma(-t + 1/2)\Gamma(t + a - 1/2)}{\Gamma(t + b - 1/2)} \left( \frac{1}{x} \right)^{-t} dt \right| a, b - 1/2, 1 \right)
\]
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where \( L' \) denotes the contour indented by 1/2 with respect to the original contour \( L \). Similarly,

\[
\pi i \frac{\Gamma(b)}{\Gamma(a)} G_{p+1,p+1}^{p+1,0} \left( \frac{1}{x} \bigg| \frac{1}{a}, b \right) = \frac{1}{2} \frac{\Gamma(b) \Gamma(s + a)}{\Gamma(s + a) \Gamma(s + b)} \left( \frac{1}{x} \right)^{-s} \int \frac{\cos(\pi s) - i \sin(\pi s)}{\Gamma(s + b)} \left( \frac{1}{x} \right)^{-s} \, ds
\]

Hence, we arrive at

\[
F_+ = \frac{\Gamma(b)}{\Gamma(a)2\pi i} \int \frac{(\cos(\pi s) - i \sin(\pi s)) \Gamma(-s) \Gamma(s + a)}{\Gamma(s + b)} \left( \frac{1}{x} \right)^{-s} \, ds
\]

\[
F_- = \frac{\Gamma(b)}{\Gamma(a)2\pi i} \int \frac{(\cos(\pi s) + i \sin(\pi s)) \Gamma(-s) \Gamma(s + a)}{\Gamma(s + b)} \left( \frac{1}{x} \right)^{-s} \, ds
\]

The ultimate Meijer’s G-function of the argument \( e^{-\pi i/x} \) can be rewritten as

\[
G_{p+1,p+1}^{p+1,1} \left( e^{-\pi i/x} \bigg| 1, \frac{1}{a} \right) = G_{p+1,p+1}^{1,p+1} \left( -x \bigg| 0, 1 - \frac{1}{a} \right).
\]

The G function on the right-hand side is a standard representation for \( G_{p+1,p+1}(a; b; x) \). Let us mention the following particular cases [1, (2.1.6) and (2.1.3)]:

- \( 1F_0(u; -z) = (1 - z)^{-u} \) and \( 2F_1(1, 1; 2; z) = -z^{-1} \log(1 - z) \). In order to apply formulas (3.1) and (3.2) to these functions we will need the reduction formulas [7, 8.4.2.8, 8.4.2.3, 8.4.6.8, 8.4.2.1]
This leads to the following relations which are also easy to verify directly \((x > 1)\)

\[
(1 - x - i0)^{-u} - (1 - x + i0)^{-u} = 2(x - 1)^{-u} \sin(\pi u),
\]

\[
(1 - x - i0)^{-u} + (1 - x + i0)^{-u} = 2(x - 1)^{-u} \cos(\pi u),
\]

\[
\frac{-\log(1 - x - i0)}{x + i0} + \frac{\log(1 - x + i0)}{x - i0} = \frac{2\pi}{x},
\]

\[
\frac{-\log(1 - x - i0)}{x + i0} - \frac{\log(1 - x + i0)}{x - i0} = -2 \log(x - 1).
\]

Note that the assumption \(-\pi < \arg(z) \leq \pi\) leads to the continuity of the functions \((1 - z)^{-u}\) and \(\log(1 - z)\) from below when \(z\) approaches the branch cut \((1, \infty)\). In a similar fashion, continuity of the function \(p + 1 F_p(a; b; z)\) from below on the branch cut \((1, \infty)\) is the consequence of \(-\pi < \arg(z) \leq \pi\) as described at the Wolfram functions site (see [23]). This is consistent with the above formulas for \(F_+\) and \(F_-\). Similar calculations can be made for more complicated Meijer’s \(G\) and Fox’s \(H\) functions, but we postpone this topic to a future publication.

In [5, Corollary 1] we observed that the function \(x \mapsto p + 1 F_p(\sigma; a; b; -z^{1/\sigma})\) is a Stieltjes function for each \(\sigma > 0\) as long as \(v_{a,b}(t) \geq 0\) on \([0, 1]\) and derived the representation

\[
p + 1 F_p(\sigma; a; b; -z) = \int_0^\infty \frac{\varphi(y) dy}{y^\sigma + z^\sigma},
\]

valid for \(\sigma \geq 2\) and \(|\arg(z)| < \pi/\sigma\). The density \(\varphi\) is given by

\[
\varphi(y) = \frac{\sigma y^{\sigma-1} \Gamma(b) \Gamma(a)}{\pi \Gamma(a)} \int_0^1 \frac{\sin(\sigma \arctan(y^{\pi/2} / \Gamma(1 + \pi)) \sqrt{1 + t^2 y^2})}{t(1 + 2 t y \cos(\pi/\sigma) + t^2 y^2)^{\sigma/2}} G_{p,0}^{p,0} \left( t \left| \begin{array}{c} b \\ a \end{array} \right. \right) dt \tag{3.6}
\]

For \(\sigma = 2\), formula (3.6) simplifies to

\[
\varphi(y) = \frac{4 \Gamma(b)}{\pi \Gamma(a)} \int_0^1 \frac{y^2}{(1 + t^2 y^2)^2} G_{p,0}^{p,0} \left( t \left| \begin{array}{c} b \\ a \end{array} \right. \right) dt \tag{3.7}
\]

by virtue of the identity \(\sin(2 \arctan(s)) = 2s / (1 + s^2)\). Below we simplify the expression for \(\varphi\) further.

**Theorem 3.2:** Suppose \(\Re(a) > 0\), then

\[
p + 1 F_p \left( a \left| \begin{array}{c} b \\ \end{array} \right. \right) - z = \frac{2(a)}{\pi(b)} \int_0^\infty \frac{t^2}{z^2 + t^2} G_{2p+1}^{2p+1} \left( \begin{array}{c} (a + 1)/2, (a + 2)/2 \\ (b + 1)/2, (b + 2)/2, 3/2 \end{array} \right) - t^2 \right) dt \tag{3.8}
\]

for \(|\arg(z)| < \pi/2\).
**Proof:** Set $\sigma = 2$ in (3.5) and calculate using the binomial expansion in (3.7)

$$
\varphi(y) = \frac{4\Gamma(b)}{\pi \Gamma(a)} \int_0^1 \frac{y^2}{(1 + t^2y^2)^2} C_{p,0}^p(t \mid \frac{b}{a}) \, dt
$$

$$
= \frac{4y^2\Gamma(b)}{\pi \Gamma(a)} \sum_{k=0}^{\infty} \frac{(2)_k(-y^2)^k}{k!} \int_0^1 C_{p,0}^p(t \mid \frac{b}{a}) t^{2k} \, dt
$$

$$
= \frac{4y^2\Gamma(b)}{\pi \Gamma(a)} \sum_{k=0}^{\infty} \frac{(2)_k(-y^2)^k}{k!} \frac{\Gamma(a + 2k + 1)}{\Gamma(b + 2k + 1)}
$$

$$
= \frac{4y^22^a\Gamma(b)\Gamma(a/2 + 1/2)\Gamma(a/2 + 1)}{\pi 2^b \Gamma(a)\Gamma(b/2 + 1/2)\Gamma(b/2 + 1)} \sum_{k=0}^{\infty} \frac{(2)_k(a/2 + 1/2)_k}{k!} \frac{(a/2 + 1/2)_k(b/2 + 1/2)_k}{(b/2 + 1/2)_k(b/2 + 1)_k} (-y^2)^k
$$

$$
= \frac{4y^2}{\pi(b)} 2^{p+1} F_{p+1} \left( \frac{2, (a + 1)/2, (a + 2)/2}{(b + 1)/2, (b + 2)/2} \mid -y^2 \right).
$$

Substituting into (3.5) we get

$$
p_{+1}F_p(a; b; -z) = p_{+2}F_{p+1}(2, a; 2, b; -z)
$$

$$
= \frac{4(a)}{\pi 2(b)} \int_0^\infty 2^{p+3} F_{2p+2} \left( \frac{2, (a + 1)/2, (a + 2)/2}{(b + 1)/2, (b + 2)/2, (2 + 1)/2, (2 + 2)/2} \bigg| -y^2 \right) \frac{y^2 \, dy}{y^2 + z^2}
$$

$$
= \frac{2(a)}{\pi(b)} \int_0^\infty 2^{p+2} F_{2p+1} \left( \frac{(a + 1)/2, (a + 2)/2}{(b + 1)/2, (b + 2)/2, 3/2} \bigg| -y^2 \right) \frac{y^2 \, dy}{y^2 + z^2}.
$$

The convergence condition $\Re(a) > 0$ is verified by (3.3).

**Remark 3.2:** Formula (3.8) is not contained in [7] and Mathematica is unable to evaluate the integral on the right-hand side.

We can deduce another simplification of (3.6) for $\sigma = 3$.

**Theorem 3.3:** Suppose $\Re(a) > 0$, then for $| \arg(z) | < \pi/3$

$$
p_{+1}F_p \left( \begin{array}{c} a \\ b \end{array} \bigg| -z \right) = \frac{9\sqrt{3}}{2\pi} \int_0^\infty \frac{\psi_1(y) - 2\psi_2(y) + 2\psi_4(y) - \psi_5(y)}{z^3 + y^3} \, dy,
$$

(3.9)

where

$$
\psi_m(y) = \frac{y^{m+2}(a)_m}{(3)_m(b)_m} 3^{3p+4} F_{3p+3} \left( \frac{3, \Delta(a + m, 3)}{\Delta(b + m, 3), \Delta(m + 3, 3)} \bigg| y^3 \right),
$$

and $\Delta(c, k) = (c/k, (c + 1)/k, \ldots, (c + k - 1)/k)$. The values of the hypergeometric function $p_{+1}F_p(t)$ for $t \in [1, \infty)$ can be taken on either side of the branch cut (but consistently for all $\psi_m$) or as the average value according to formula (3.2). In either case, the numerator of the integrand in (3.9) takes real values.
Proof: First, we treat the case of $p_+ F_p(3, a; b; -z)$, where $a$ and $b$ both have $p$ components. Taking $\sigma = 3$ in (3.5) we want to calculate $\varphi(y)$ in (3.6). Using the formulas [10, (18.5.2)]

$$\sin(n\theta) = \sin(\theta)U_{n-1}(\cos(\theta))$$ \quad and \quad $$\cos(\arctan(x)) = (1 + x^2)^{-1/2},$$

where $U_n$ denotes the Chebyshev polynomial of the second kind, and [10, (18.5.15)] we get

$$\sin(3 \arctan(x)) = \frac{x(3 - x^2)}{(1 + x^2)^{3/2}}.$$

Tedious but elementary calculations then yield

$$\frac{\sin(3 \arctan(\frac{ty\sin(\pi/3)}{1 + ty\cos(\pi/3)}))}{(1 + 2ty \cos(\pi/3) + t^2y^2)^{3/2}} = \frac{3\sqrt{3}ty(1 + ty)}{2(1 + ty + t^2y^2)^3} = \frac{3\sqrt{3}ty(1 - 2ty + 2t^3y^3 - t^4y^4)}{2(1 - t^3y^3)^3}$$

(3.10)

Applying the binomial expansion to $$(1 - t^3y^3)^{-3}$$ and integrating termwise we obtain

$$\varphi(y) = \frac{9\sqrt{3}y^2\Gamma(b)}{2\pi\Gamma(a)} \int_0^1 ty(1 - 2ty + 2t^3y^3 - t^4y^4) \sum_{k=0}^\infty \frac{(3)_k}{k!} (ty)^k G^{p,0}_{p,p}(t \bigg| \frac{b-1}{a-1}) dt$$

$$= \frac{9\sqrt{3}y^2\Gamma(b)}{2\pi\Gamma(a)} (\varphi_1(y) - 2\varphi_2(y) + 2\varphi_4(y) - \varphi_5(y)),$$

where

$$\varphi_m(y) = \int_0^1 \sum_{k=0}^\infty \frac{(3)_k}{k!} (ty)^k G^{p,0}_{p,p}(t \bigg| \frac{b-1}{a-1}) dt$$

$$= \sum_{k=0}^\infty \frac{(3)_k}{k!} y^{3k+m} \int_0^1 t^{3k+m-1} G^{p,0}_{p,p}(t \bigg| \frac{b}{a}) dt$$

$$= \sum_{k=0}^\infty \frac{(3)_k}{k!} y^{3k+m} \int_0^1 t^{3k+m-1} G^{p,0}_{p,p}(t \bigg| \frac{b}{a}) dt$$

$$= \sum_{k=0}^\infty \frac{(3)_k}{k!} y^{3k+m} \Gamma(a + 3k + m) \Gamma(b + 3k + m) = 3\sum_{i=1}^p (a_i - b_i) \sum_{k=0}^\infty \frac{(3)_k}{k!} y^{3k+m}$$

$$\times \frac{\Gamma((a + m)/3 + k)\Gamma((a + m + 1)/3 + k)\Gamma((a + m + 2)/3 + k)}{\Gamma((b + m)/3 + k)\Gamma((b + m + 1)/3 + k)\Gamma((b + m + 2)/3 + k)}$$

$$= y^m \frac{\Gamma(a + m)}{\Gamma(b + m)^{3p+1} F_{3p} \left( \frac{3, (a + m)/3, (a + m + 1)/3, (a + m + 2)/3}{(b + m)/3, (b + m + 1)/3, (b + m + 2)/3} \mid y^3 \right)}.$$

According to (3.5) the function $\varphi(y)$ is the density in the representation of $p_+ F_p(3, a; b; -z)$. For the general case just write $p_+ F_p(a; b; -z) = p_{+2} F_{p+1}(3, a; 3, b; -z)$ and apply (3.5) with $b$ substituted by $(3, b)$. This leads immediately to formula (3.9). To verify the reality of the numerator in (3.9) note that we can rewrite $\varphi(y)$ using the first formula
in (3.10)

\[
\varphi(y) = \frac{9\sqrt{3}y^3\Gamma(b)}{2\pi\Gamma(a)} \int_0^1 \frac{1 + ty}{(1 + ty + t^2y^2)^3} G_{p,0}^{p,0} \left( t \left| \begin{array}{l} b \\ a \end{array} \right. \right) dt.
\]

The integral here is well-defined and real for all \( y > 0 \). The convergence condition \( \Re(a) > 0 \) can be verified by application of (3.3) to each \( \psi_m \) in the integrand.

**Remark 3.3:** The above formula for \( \varphi(y) \) permits derivation of another expression for this function as a double hypergeometric series. Indeed, writing

\[
\frac{1 + ty}{(1 + ty + t^2y^2)^3} = \frac{1 + ty}{(1 + ty(1 + ty))^3} = \sum_{k=0}^{\infty} (-1)^k \frac{(3)_k}{k!} (ty)^k (1 + ty)^{k+1}
\]

we can substitute this into the above integral and integrate termwise to get

\[
\varphi(y) = \frac{9\sqrt{3}y^3\Gamma(b)}{2\pi\Gamma(a)} \sum_{k=0}^{\infty} (-1)^k \frac{(3)_k}{k!} y^{n+k} \sum_{n=0}^{k+1} \binom{k+1}{n} \int_0^1 t^{n+k} G_{p,0}^{p,0} \left( t \left| \begin{array}{l} b \\ a \end{array} \right. \right) dt
\]

**Remark 3.4:** One can verify the continuity of the numerator \( \psi_1(y) = 2\psi_2(y) + 2\psi_4(y) - \psi_5(y) \) of the integrand in (3.8) in the neighbourhood of the branch cut \([1, \infty)\) by direct application of Theorem 3.1. Indeed, according to (3.1) the jump of \( \psi_m(y) \) over the ray \( y > 1 \) equals

\[
2\pi i y^{m+2} \frac{(a)_m \Gamma(\Delta(b + m, 3), \Delta(m + 3, 3))}{(3)_m(b)_m \Gamma(3, \Delta(a + m, 3))} \times G_{3p+4,0}^{3p+4,3p+4} \left( \frac{1}{y^3} \left| \begin{array}{l} 1, \Delta(b + m, 3), \Delta(m + 3, 3) \\ \Delta(a + m, 3) \end{array} \right. \right).
\]

By the Gauss multiplication formula \( \Gamma(\Delta(x + m, 3))/(x)_m = 2\pi \Gamma(x)^3 x^{1/2-x-m} \), so that the expression

\[
\frac{(a)_m \Gamma(\Delta(b + m, 3), \Delta(m + 3, 3))}{(3)_m(b)_m \Gamma(3, \Delta(a + m, 3))}
\]
is independent of \( m \). Hence, the jump of \( \psi_1(y) - 2 \psi_2(y) + 2 \psi_4(y) - \psi_5(y) \) is equal to zero if

\[
G^{3p+4,0}_{3p+4,3p+4} \left( \frac{1}{y^3} \bigg| \begin{array}{c} 1, \Delta(b + 1, 3), \Delta(4, 3) \\ 3, \Delta(a + 1, 3) \end{array} \right)
- 2yG^{3p+4,0}_{3p+4,3p+4} \left( \frac{1}{y^3} \bigg| \begin{array}{c} 1, \Delta(b + 2, 3), \Delta(5, 3) \\ 3, \Delta(a + 2, 3) \end{array} \right)
+ 2y^3G^{3p+4,0}_{3p+4,3p+4} \left( \frac{1}{y^3} \bigg| \begin{array}{c} 1, \Delta(b + 4, 3), \Delta(7, 3) \\ 3, \Delta(a + 4, 3) \end{array} \right)
- y^4G^{3p+4,0}_{3p+4,3p+4} \left( \frac{1}{y^3} \bigg| \begin{array}{c} 1, \Delta(b + 5, 3), \Delta(8, 3) \\ 3, \Delta(a + 5, 3) \end{array} \right) = 0.
\]

Writing \( y^{-3} = t \) and using the shifting property of \( G \) function \([7, 8.2.2.15]\) the last formula takes the form

\[
G^{3p+4,0}_{5p+4,3p+4} \left( t \bigg| \begin{array}{c} 1, \Delta(b + 1, 3), \Delta(4, 3) \\ 3, \Delta(a + 1, 3) \end{array} \right)
- 2G^{3p+4,0}_{5p+4,3p+4} \left( t \bigg| \begin{array}{c} 2/3, \Delta(b + 1, 3), \Delta(4, 3) \\ 8/3, \Delta(a + 1, 3) \end{array} \right)
+ 2G^{3p+4,0}_{5p+4,3p+4} \left( t \bigg| \begin{array}{c} 0, \Delta(b + 1, 3), \Delta(4, 3) \\ 2, \Delta(a + 1, 3) \end{array} \right)
- G^{3p+4,0}_{3p+4,3p+4} \left( t \bigg| \begin{array}{c} -1/3, \Delta(b + 1, 3), \Delta(4, 3) \\ 5/3, \Delta(a + 1, 3) \end{array} \right) = 0.
\]

To verify this identity write the definition of \( G \) function \([7, 8.2.1.1]\) and collect the integrands under single integral sign to get

\[
\frac{\Gamma(s + 3)}{\Gamma(s + 1)} - 2 \frac{\Gamma(s + 8/3)}{\Gamma(s + 2/3)} + 2 \frac{\Gamma(s + 2)}{\Gamma(s)} - \frac{\Gamma(s + 5/3)}{\Gamma(s - 1/3)} = 0,
\]

which is true by the shifting property \( \Gamma(z + 1) = z\Gamma(z) \).

The following identity reduces to Newton–Leibnitz formula under substitution \( u = 1/(x + t) \):

\[
\phi(1/x) = \phi_0 + \int_0^{\infty} \frac{\phi'(1/(x + t))}{(x + t)^2} \, dt,
\]

where \( \phi'(z) = d\phi(z)/dz \). It is valid for differentiable \( \phi \) under convergence of the integral in \((3.11)\). Applying \((3.11)\) to the generalized hypergeometric function we obtain for \( p \leq q \)

\[
pFq \left( \begin{array}{c} a \\ b \end{array} \bigg| \frac{1}{x} \right) = 1 + \frac{a}{b} \int_0^{\infty} (x + t)^{-2} pFq \left( \begin{array}{c} a + 1 \\ b + 1 \end{array} \bigg| \frac{1}{x + t} \right) \, dt,
\]

where we have used the standard formula for the derivative of \( pFq \) \([10, (16.3.1)]\). This simple relation is not contained in \([7]\), but \texttt{Mathematica} is able to evaluate the integral on the right-hand side.
A straightforward consequence of a result due to Thale [24] observed by us in [5, Theorem 13] is that the generalized hypergeometric function $p_{+1}F_p(z)$ is univalent in $\Re(z) < 1$ as long as it belongs to the Stieltjes class. Univalent functions in the unit disk satisfy a number of distortion theorems outlined, for instance, in [25, Theorem 7.1]. Below we illustrate the application of these theorems to the generalized hypergeometric function. We will need the following inequalities satisfied by the functions $f$ holomorphic and univalent in the unit disk and normalized by $f(0) = f'(0) - 1 = 0$:

\[
\frac{|w|}{(1 + |w|)^2} \leq |f(w)| \leq \frac{|w|}{(1 - |w|)^2},
\]

(3.12)

\[
\frac{1 - |w|}{(1 + |w|)^3} \leq |f'(w)| \leq \frac{1 + |w|}{(1 - |w|)^3},
\]

(3.13)

\[
\frac{1 - |w|}{|w|(1 + |w|)} \leq \frac{|f'(w)|}{|f(w)|} \leq \frac{1 + |w|}{|w|(1 - |w|)},
\]

(3.14)

**Theorem 3.4:** Suppose $a = (a_1, \ldots, a_{p+1})$ and $b = (b_1, \ldots, b_p)$ are positive vectors such that, $0 < a_1 \leq 1$ and $v_{a_1}, b(t) \geq 0$ on $[0, 1]$, where $v_{a_1}, b$ is defined in (1.4). Then the following inequalities hold in the half-plane $\Re(z) < 1$:

\[
\frac{2(a)|z - 2||z|}{(b)(|z - 2| + |z|)^3} \leq |p_{+1}F_p(a; b; z) - 1| \leq \frac{2(a)|z - 2||z|}{(b)(|z - 2| - |z|)^3},
\]

(3.15)

\[
\frac{4(|z - 2| - |z|)}{(|z - 2| + |z|)^3} \leq |p_{+1}F_p(a + 1; b + 1; z)| \leq \frac{4(|z - 2| + |z|)}{(|z - 2| - |z|)^3}
\]

(3.16)

and

\[
\frac{2(b)(|z - 2| - |z|)}{(a)|z||z - 2|(|z - 2| + |z|)} \leq \left| \frac{p_{+1}F_p(a + 1; b + 1; z) - 1}{p_{+1}F_p(a; b; z)} \right| \leq \frac{2(b)(|z - 2| + |z|)}{(a)|z||z - 2|(|z - 2| - |z|)}.
\]

(3.17)

**Proof:** According to [5, Theorem 13] the function $z \mapsto p_{+1}F_p(a; b; z)$ is univalent in the half-plane $\Re(z) < 1$ under the hypotheses of the theorem. The Möbius map $z = z(w) = 2w/(w - 1)$ effects a biholomorphic bijection between the unit disk $|w| < 1$ and the half-plane $\Re(z) < 1$. Hence, inequalities (3.12)–(3.14) hold for the function

\[
f(w) = \frac{(b)}{2(a)} \left( 1 - p_{+1}F_p \left( \frac{a}{b} \bigg| z(w) \right) \right)
\]

satisfying $f(0) = f'(0) - 1 = 0$. In view of the derivative formula for the generalized hypergeometric function and substituting $w = z/(z - 2)$ we obtain (3.15)–(3.17).
Remark 3.5: It is a classical fact that equality is attained in (3.12)–(3.14) for the rotations of the Koebe function \( K(w) = w/(1 + w)^2 \), see [25, Theorem 7.1]. The hypergeometric expression for \( K(w(z)) \) is given by

\[
K(w) = K\left( \frac{z}{z-2} \right) = \frac{z(z-2)}{4(1-z)^2} = \frac{1}{4} \left( 1 - \frac{1}{(1-z)^2} \right) = \frac{1}{4} \left( 1 - \frac{1}{(1+z)^2} \right).
\]

Comparing this expression with \( f(w) \) defined in the proof of Theorem 3.4, we see that the function \( g(z) = \frac{1}{2} \binom{1}{2} ; 1 \frac{1}{z} \) is extremal in Theorem 3.4, so that equality is attained in (3.15)–(3.17) for this function and some values of \( z \). On the other hand, \( g(z) \) clearly violates the conditions of Theorem 3.4 as \( v_{1,2} (t) = t(t-1) < 0 \) for \( t \in (0,1) \). Since \( g(z) \) is univalent, it is clear that conditions of Theorem 3.4 are far from necessary for univalence of \( p+1 \binom{a}{b} ; b ; z \) in \( \Re z < 1 \) and for the validity of (3.15)–(3.17). Hence, there is a room for relaxing these conditions.

4. A curious integral evaluation

In the final section of the paper we present a curious representation for a finite sum of product ratios of gamma functions as an integral of Meijer’s G function. If the dimension \( p = 1 \), the integrand is expressed via the Gauss hypergeometric function as stated in Corollary 4.1.

Theorem 4.1: Suppose \( a, b \in \mathbb{C}^p \) are such that \( 0 < \Re (a_j) < \Re (\sum_{j=1}^{p} b_j) \) for \( j = 1, \ldots, p, \alpha, \beta > 0 \) and \( m \in \mathbb{N}_0 \). Then the following identity holds:

\[
\sum_{k=0}^{m} \left\{ \frac{\Gamma(a + k)\Gamma(a + \alpha + \beta + m - k)}{\Gamma(b + k)\Gamma(b + \alpha + \beta + m - k)} \right\} = \int_{0}^{1} \frac{1}{G_{2p-2p}^{p-p}(x)} \left( \frac{1 - x^{m+1}(1 - x^\alpha)(1 - x^\beta)}{x^{\alpha+\beta+1}(1 - x)} \right) dx. \tag{4.1}
\]

Proof: Suppose \( m+n = p \). We will write \( b = (bt, bb) \), where \( bt \) stands for the first \( m \) components of \( b \) (located in the numerator of the integrand, hence \( bt \) for \( b \)-top) and \( bb \) for the last \( q-m \) components (located in the denominator of the integrand, hence \( bb \) for \( b \)-bottom). Similarly, \( a = (at, ab) \) with \( n \) components in \( at \) and \( p-n \) in \( ab \). Then, according to [7, 8.2.2.7, 26], Meijer’s G function \( G_{p,p}^{m,n}(z) \) is piecewise analytic with discontinuity on the unit circle \( |z| = 1 \). In this case, combining [11, Theorem 12.5.1] with [11, Theorem 12.5.2] or [7, 8.2.2.3] with [7, 8.2.2.4], we can write for \( x > 0 \)

\[
G_{p,p}^{m,n}(x) = H(1 - x) \sum_{j=1}^{m} A_j x^{b_j} \binom{1}{p} F_{p-1} \left( \frac{1 - a + b_j}{1 - b_{[j]} + b_j} \right) x + H(x - 1) \sum_{j=1}^{n} B_j x^{a_j} \binom{1}{p} F_{p-1} \left( \frac{1 + b - a_j}{1 + a_{[j]} - a_j} \right) \frac{1}{x}. \tag{4.2}
\]
where

\[
A_j = \frac{\Gamma(bt_j - b_j)\Gamma(1 - at + bj)}{\Gamma(1 - bb + bj)\Gamma(ab - b_j)}, \quad B_j = \frac{\Gamma(a_j - at_{j})\Gamma(1 + bt - a_j)}{\Gamma(1 + ab - a_j)\Gamma(ab - b_j)}
\]

and \(H(t) = \partial_t \max\{t, 0\}\) is the Heaviside function. Next, according to [7, 2.24.2.1] (see also [27, Theorem 2.2 and 3.3])

\[
\frac{\Gamma(a + k)\Gamma(a + \alpha + \beta + m - k)}{\Gamma(b + k)\Gamma(b + \alpha + \beta + m - k)} = \int_0^\infty x^{k-1} G_{2p,2p}^{p,p} \left( x \left| \begin{array}{c} 1 - a - \alpha - \beta - m, b + \alpha \\ a + \alpha, 1 - b - \beta - m \end{array} \right. \right) dx
\]

where in the last equality we have used the shifting property [7, 8.2.2.15]. Similarly

\[
\frac{\Gamma(a + \alpha + k)\Gamma(a + \beta + m - k)}{\Gamma(b + \alpha + k)\Gamma(b + \beta + m - k)} = \int_0^\infty x^{k-\beta} G_{2p,2p}^{p,p} \left( x \left| \begin{array}{c} 1 - a - \beta - m, b + \alpha \\ a + \alpha, 1 - b - \beta - m \end{array} \right. \right) dx
\]

Note that the \(G\) functions in the integrand are the same, so that we can also write the left-hand side of (4.1) as

\[
\text{LHS of (4.1)} = \int_0^\infty x^{k-\beta} (x^{-\alpha} - 1) G_{2p,2p}^{p,p} \left( x \left| \begin{array}{c} 1 - a - m, b + \alpha + \beta \\ a + \alpha + \beta, 1 - b - m \end{array} \right. \right) dx. \quad (4.3)
\]

However, the function in the integrand is piecewise analytic, so essentially we are dealing with a sum of two different integrals. To reduce it to a single integral we apply (4.2) to the \(G\) function in the integrand

\[
G_{2p,2p}^{p,p} \left( x \left| \begin{array}{c} 1 - a - m, b + \alpha + \beta \\ a + \alpha + \beta, 1 - b - m \end{array} \right. \right)
\]

\[
= H(1 - x) \sum_{j=1}^{p} M_j x^{a_j+\alpha+\beta} \binom{a + m + \alpha + \beta + a_j, 1 - b + a_j}{1 - a_{j}, a_j, b + m + \alpha + \beta + a_j} (1 - x) + H(x - 1) \sum_{j=1}^{p} M_j x^{-a_j-m} \binom{a + m + \alpha + \beta + a_j, 1 - b + a_j}{1 - a_{j}, a_j, b + m + \alpha + \beta + a_j} \left( \frac{1}{x} \right),
\]

(4.4)

where

\[
M_j = \frac{\Gamma(a_{j} - a_j)\Gamma(a + m + \alpha + \beta + a_j)}{\Gamma(b + m + \alpha + \beta + a_j)\Gamma(b - a_j)}.
\]

(4.5)

Note the surprising fact that both the constants and the hypergeometric functions in the two sums are exactly the same. Next, substitute this expansion into the integrand in (4.3),
carry out change of variable $x \rightarrow 1/x$ in the second integral and add up the resulting integrals to get

$$\text{LHS of (4.1)} = \int_0^1 \left[ \sum_{j=1}^{P} M_j x^{a_j-1} \right] \times 2_p F_{2p-1} \left( \begin{array}{c} a + a_j + \alpha + \beta + m, 1 - b + a_j \\ 1 - a_{[j]} + a_j, b + a_j + \alpha + \beta + m \end{array} \right) \frac{(1 - x^{m+1})(1 - x^{a})(1 - x^{\beta})}{1 - x} \ dx.$$  

(4.6)

In order to obtain (4.1) it remains to observe that the expression in brackets equals the left-hand side of (4.4) times $H(1 - x)x^{-a-\beta}$. Conditions for convergence follow by applying the expansion [22, (2.19)] of $2_p F_{2p-1}(z)$ in the neighbourhood of unity (due to Nørlund and Bühring) to the first sum in (4.4).

Taking $p = 1$ in the above theorem we immediately get the next.

**Corollary 4.1:** Suppose $\Re(b + 1) > \Re(a) > 0$, $\alpha, \beta > 0$ and $m \in \mathbb{N}_0$. Then the following identity holds:

$$\sum_{k=0}^{m} \left\{ \frac{\Gamma(a + k)\Gamma(a + \alpha + \beta + m - k)}{\Gamma(b + k)\Gamma(b + \alpha + \beta + m - k)} - \frac{\Gamma(a + \alpha + k)\Gamma(a + \beta + m - k)}{\Gamma(b + \alpha + k)\Gamma(b + \beta + m - k)} \right\}$$

$$= \frac{\Gamma(2a + \alpha + \beta + m)}{\Gamma(a + b + \alpha + \beta + m)\Gamma(b - a)} \int_0^1 x^{a-1}(1 - x^{m+1})(1 - x^{a})(1 - x^{\beta})$$

$$\times 2_F_1 \left( \begin{array}{c} 2a + \alpha + \beta + m, 1 + a - b \\ a + b + \alpha + \beta + m \end{array} \right) \frac{dx}{1 - x}. \quad (4.7)$$

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**ORCID**

D. B. Karp [http://orcid.org/0000-0001-8206-3539]

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